from stable to unstable regions are those of period T and $2T$ (2π and 4π in this case). The monodromy matrix **C** of Eq. (7) is such that

$$\det\mathbf{C} = \exp\left\{\int_0^{2\pi} \text{trace}\mathbf{A}(\tau)d\tau\right\} = \exp\{-2\pi/\Pi_1\}$$

so that if λ_1, λ_2 are the eigenvalues of C then

$$\lambda_1\lambda_2 = \exp\{-2\pi/\Pi_1\} = \rho^2 \text{ (say)} \tag{12}$$

If λ_1, λ_2 are complex conjugates then it follows from Eq. (12) that they lie on a circle of radius ρ so that complex roots cannot have modulii unity since this would imply $\rho = 1$ which is only possible if Π_1 is infinite. Thus on the transition boundary there must exist a real root having the value of $+1$ or -1 (note that real roots λ_1,λ_2 are inverse points with respect to the circle of radius ρ). It follows from Eq. (11) that the transition boundary is characterized by a solution of period T or $2T$.

On substituting a Fourier series, with undetermined coefficients, of period 4π in Eqs. (7) and balancing like terms, it can be shown, by induction, that the corresponding Hill determinants are sums of squares and therefore cannot be zero for any values of the parameters. This has been verified by analogue computer simulation and by the results of the Floquet theory analysis.

In the case of the harmonic solution, of period 2π, substituting the Fourier series

$$\xi_1 = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\tau + b_n \sin n\tau)$$

into Eqs. (7) [the corresponding series for ξ_2 being obtained by the second equation of (7)] and balancing the terms leads to two distinct sets of linear homogeneous algebraic equations for the coefficients (a_{2n},b_{2n}), (a_{2n+1},b_{2n+1}), ($n = 0,1,2$, etc.), respectively. The corresponding Hill determinants of order r, in each case, are polynomials of order r in Π_2 having coefficients which are functions of Π_1. For a particular r these polynomials are solved for a range of values of Π_2 and the zeros plotted to obtain the transition boundary in the parameter space. The value of r is then increased and the corresponding Hill determinants solved until a convergent set of boundaries are obtained. It is found that for $\Pi_1 > 1.5$, where the enveloping boundary is continuous, consideration of fifth order Hill determinants is sufficient but for $\Pi_1 < 1.5$ where the enveloping boundary is discontinuous, the method is not found to be very satisfactory. Hill determinants of order eleven have to be considered before a true picture begins to emerge and the order has to be increased still further before an enveloping boundary is obtained to a satisfactory degree of accuracy. For this problem the region $\Pi_1 < 1.5$ is important since it is the most likely range of application in practice.

Conclusions

In this Note the stability regions in nondimensional space have been obtained for a first-order controllable gain model reference adaptive control system, and the results illustrate the complexity of the question of stability when dealing with such systems.

Both a numerical implementation of Floquet analysis and the infinite determinant method of analyzing linear differential equations with periodic coefficients have been employed. The infinite determinant approach was not found to be very satisfactory in the region of parameter space where the stability boundaries are complex in nature. Since, when dealing with linear differential equations with periodic coefficients, complex stability boundaries frequently occur, it throws some doubt on the performance of the method in general. Although the Floquet analysis involved investigating the eigenvalues of the monodromy matrix at a network of points in parameter space the results obtained were far more satisfactory, and for the particular problem considered the computation time was less. For higher order systems the use of the Faddeev algorithm and the Jury procedure would further reduce computational time when using the Floquet approach.

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Some Considerations of a Simplified Velocity Spectrum Relation for Isotropic Turbulence

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THE "three-dimensional" velocity spectrum function, $E(k)$, defined such that

$$\frac{3}{2}\hat{u}^2 = \int_0^\infty E(k)dk \tag{1}$$

($\hat{u}$ = rms velocity fluctuation level, k = wave number), is of interest in both theoretical and practical studies of turbulence phenomena. A simple form for $E(k)$ was proposed by von Kármán[1]

$$k_eE(k)/\hat{u}^2 \propto (k/k_e)^4[1 + (k/k_e)^2]^{-17/6} \tag{2}$$

(k_e = energy-containing wave number, as an "interpolation" formula joining the range $k \approx 0(E(k) \propto k^4)$ to the inertial subrange, wherein $E(k) \propto k^{-5/3}$ for $k \gg k_e$ in Eq. (2).

At high-wave numbers, of the order of the Kolmogoroff wave number $k_K \doteq L_K^{-1}$, where

$$L_K = (\nu^3/\Phi)^{1/4} \tag{3}$$

the spectrum function is "cut-off" by viscous effects. The inadequacy of Eq. (2) for this wave number range is reflected

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by the divergence of the dissipation integral

$$\Phi = 2\nu \int_0^\infty k^2 E(k) dk \quad (4)$$

for this case.

It is the purpose of this Note to furnish Eq. (2) with a simple (exponential) viscous cut-off, and to examine some consequences of the resultant, more general expression for $E(k)$.

Spectral Behavior in the Viscous Range

A model for the migration of spectral content from lower to higher wave numbers within the viscous range has been proposed by Townsend.[2] Its essential feature is a uniform straining action by large-scale eddies on the randomly-oriented vortex sheets of the small-scale structure; the resultant velocity spectrum prediction is in good agreement with data.[2]

Another approach has been given by Corrsin,[3] drawing on ideas of spectral-content cascading from lower to higher wave numbers originated by Onsager. For present purposes, this model states that spectral flux $F(k)$ changes with increasing wave number at a rate equal to the local viscous dissipation rate at the wave number k, i.e.,

$$dF(k)/dk = -2\nu k^2 E(k) \quad (5)$$

For $F(k)$, we follow Corrsin[3]

$$F(k) = kE(k)/\tau(k) \quad (6)$$

and for the characteristic cascade time, $\tau(k)$, we use Townsend's uniform strain rate[2] in a slightly more general form

$$\tau(k) = q(\nu/\Phi)^{1/2} = \text{const} \quad (7)$$

where q is an empirical constant. Letting $\xi = kL_K$, combination of Eqs. (5–7) yields

$$E(\xi)/U_K{}^2 L_K = A_c \xi^{-1} \exp(-q\xi^2) \quad (8)$$

where A_c is a constant of order unity and U_K is the Kolmogoroff velocity

$$U_K = (\nu\Phi)^{1/4} \quad (9)$$

The form of Eq. (8) agrees with the "universal equilibrium" postulate of Kolmogoroff for this wave number range.

A comparison with Townsend's result can be made by first noting that the one-dimensional energy spectrum function, $E_1(k_1)$, is related to $E(k)$ by[4]

$$E_1(k_1) = \int_{k_1}^\infty k^{-1} E(k)[1 - (k_1/k)^2] dk \quad (10)$$

Then, with $\xi_1 = k_1 L_K$ and $s = k_1/k$, Eqs. (8 and 10) give

$$E_1(\xi_1)/U_K{}^2 L_K = A_c \xi_1{}^{-1} \int_0^1 (1 - s^2) \exp[-q(\xi_1/s)^2] ds \quad (11)$$

This differs from Townsend's vortex-sheet model[2] in that $\xi_1{}^{-1}$ instead of $\xi_1{}^{-2}$ appears before the integral, and q instead of 2 is in the exponential.

Figure 1 shows Eq. (11) multiplied by $\xi_1{}^6$ to emphasize the dissipation range, as suggested by Townsend[2]; A_c was chosen to give a peak amplitude of 0.003. A value $q = 2.7$ was chosen as a "best" fit to the data. It is worth noting that the value of the "strain-rate constant," $q = 2.7$, falls within the range determined by Grant et al.[5] from curve fits to temperature fluctuation spectra in a tidal channel.

Generalized Expression for $E(k)$

The foregoing considerations show that an adequate description of $E(k)$ in the viscous range is given by the form

$$E(k) \propto k^{-n} \exp[-q(kL_K)^2] \quad (12)$$

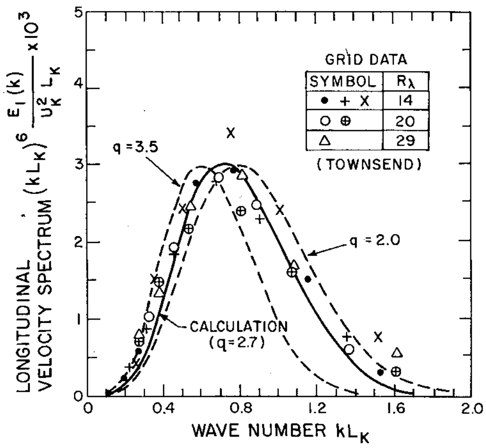

Fig. 1 Calculations based on Corrsin's cascade model, with q = 2.0, 2.7, and 3.5 (q = 2.7 taken as "best" fit), compared to Townsend's data given in Ref. 2.

n may lie between 1 and 2. An essential feature seems to be the exponential cut-off.

Therefore, we propose to modify Eq. (2) by the addition of an exponential factor as given in Eq. (12), whereby the three-dimensional energy spectrum function can be written as

$$E(k)/U_K{}^2 L_K = A(k/k_e)^4 \times [1 + (k/k_e)^2]^{-17/6} \exp(-qk^2 L_K{}^2) \quad (13)$$

valid for all wave numbers.

Some Results

We now calculate absolute spectral levels along with certain "universal constants" of interest in turbulence studies.

To relate more directly to experimental work, we introduce the turbulent Reynolds number

$$R_\lambda \doteq \hat{u}\lambda_g/\nu = 15^{1/2}(\hat{u}/U_K)^2 \quad (14)$$

where the Taylor microscale, λ_g, has been written in terms of Φ, $\hat{u}$, and ν by use of the isotropic relation[6]

$$\Phi = 15\nu(\hat{u}/\lambda_g)^2 \quad (15)$$

and Eq. (9) then used.

Letting $x = kL_K$, and the parameter

$$b = (k_e L_K)^{-1} \quad (16)$$

we define a function G as

$$G \equiv G(b,x) = [1 + (bx)^2]^{-17/6} \exp(-qx^2) \quad (17)$$

and the integrals

$$F_1 = \int_0^\infty x^4 G dx \quad F_2 = \int_0^\infty x^6 G dx \quad (18)$$

Then the normalization and dissipation relations, Eqs. (1) and (4), become, in conjunction with Eqs. (3) and (9),

$$\hat{u}/U_K = (F_1/3F_2)^{1/2} \text{ and } A = (2F_2 b^4)^{-1} \quad (19)$$

For a given q, the integrals F_1 and F_2, thus A and the ratio $\hat{u}/U_K$, are functions only of b or, equivalently, R_λ through Eq. (14). Consequently, with $k/k_e \equiv b(kL_K)$ by definition, the right-hand side of Eq. (13) is determined as a function of kL_K; Eqs. (10) and (13) are equivalent in this respect. Because of the ease of programing these relations for the electronic computer, we proceed directly to the numerical results.

Figure 2 presents calculations of the one-dimensional velocity spectrum for various R_λ and $q = 2.7$. Included in the plot are data of Grant et al.[7] from a tidal channel and Gibson et al.[8] from a sphere wake in a water tunnel. Similar

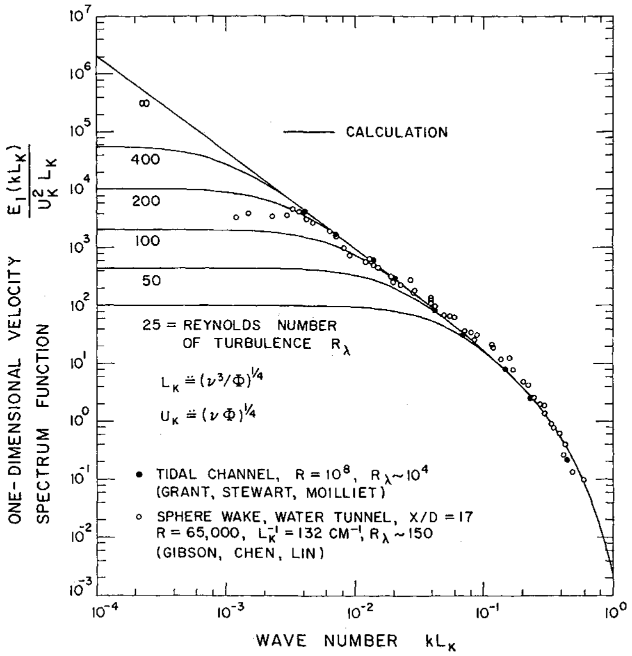

Fig. 2 One-dimensional velocity spectrum function vs normalized wave number kL_K, with Reynolds number R_λ as parameter. Tidal channel and sphere wake data from Refs. 7 and 8, respectively.

good agreement with measurements is found in the water tunnel grid data of Gibson and Schwarz.[9]

Development of the inertial subrange with increasing R_λ is clearly seen. In passing, attention is called to a similar plot given by Uberoi and Freymuth.[10] Their expression for $E_1(k)$ is presented as a curve fit, but apparently without physical justification. Furthermore, their $E_1(k)$ cannot be converted to $E(k)$ (by differentiation) in the vicinity of $k = 0$ because of their choice of form.

Of interest in turbulence studies is the "dissipation coefficient" defined by Batchelor as[11]

$$K_B \doteq -(L_f/\hat{u}^3)d\hat{u}^2/dt \equiv 2\Phi L_f(3\hat{u}^3)^{-1} = (2/3)(U_K/\hat{u})^3 b k_e L_f \quad (20)$$

where Eqs. (3) and (9) have been used for Φ.

The longitudinal velocity macroscale L_f, which is the measurable quantity rather than k_e, is related to the latter through the relations[4]

$$L_f = \pi E_1(0)(2\hat{u}^2)^{-1} = \pi(2\hat{u}^2)^{-1}\int_0^\infty k^{-1}E(k)dk \quad (21)$$

or, using Eqs. (13, 16, 17, and 19),

$$k_e L_f = 3\pi(4bF_1)^{-1}\int_0^\infty x^3 G dx \quad (22)$$

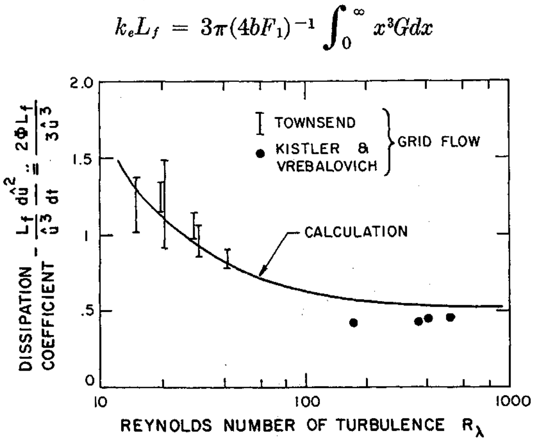

Fig. 3 Dissipation coefficient vs Reynolds number. Measurements are taken from Townsend as reported by Batchelor,[11] and from Kistler and Vrebalovich.[12]

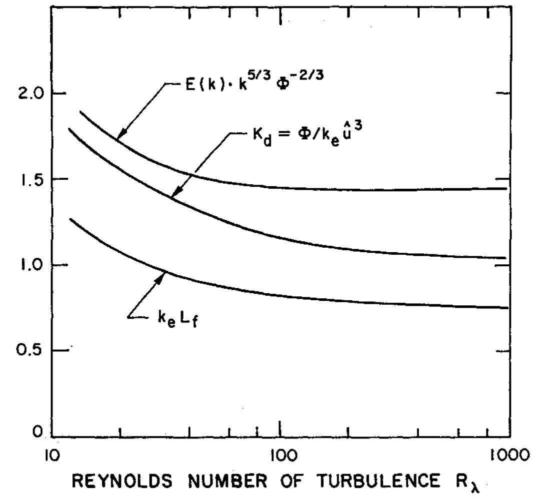

Fig. 4 Calculations of some turbulence "constants" vs Reynolds number as described in the text.

In Fig. 3 is plotted the calculated dissipation coefficient K_B vs R_λ. Comparison is made to data of Townsend as reported by Batchelor,[11] where the bars indicate maximum and minimum values obtained from all stations downstream of the grid at a given R_λ. Also shown are experimental values inferred from data of Kistler and Vrebalovich[12] at much higher R_λ. The trend and levels of the calculation are well-supported by the data.

Figure 4 presents calculations of certain "constants" frequently used in turbulence studies as a function of R_λ.

The dissipation constant K_d, defined by the relation[11]

$$\Phi = K_d k_e \hat{u}^3 \quad (23)$$

is directly analogous to K_B. However, the product $k_e L_f$ varies, as shown in Fig. 4, in such a way as to make K_d somewhat less sensitive than K_B to R_λ, as is seen.

As calculated by Hinze[4] from the von Karman interpolation formula, $k_e L_f$ has a value 0.75. This is seen in Fig. 4 to be reached at about $R_\lambda \gtrsim 700$, which would in a sense define the beginning of the "infinite" Reynolds number regime.

The inertial subrange "constant"

$$E(k) = \text{const}\cdot\Phi^{2/3}k^{-5/3} \quad (24)$$

is also polotted in Fig. 4 vs R_λ. Its asymptotic value, 1.44, is representative of data obtained from various sources.[7,9,10] According to present calculation, the asymptotic regime for this "constant" begins at values of R_λ near 100.

Concluding Remarks

A simple form for the isotropic velocity fluctuation spectrum, Eq. (13), has yielded a number of results consistent with turbulent flow measurements taken under a wide range of conditions. Two parameters specify the spectrum, one an absolute "mean strain-rate constant," the other the (variable) Reynolds number R_λ; a value for the former parameter was obtained from low-speed grid flow data. Other approaches have appeared, notably the more mathematical one of Shkarofsky.[13] However, the intent of this paper is to provide a formulation more usable for calculational purposes. Its interesting features (e.g., use of Corrsin's cascade model, the exponential nature of the viscous cut-off region, comparisons with experiments) are not necessarily meant to be taken "seriously" at this time; at the very least, further experimental and theoretical developments are needed if they are ever to be tied into a larger framework.

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Three-Dimensional Stagnation-Point Heat Transfer in Equilibrium Air Flows

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Nomenclature

C = $\rho\mu/(\rho\mu)_\infty$
f = stream function such that $\partial f/\partial\eta = (u_i/u_{i\infty})$; $i = 1,2$
g = enthalpy function $= h/h_\infty$
h = enthalpy
K = transverse to principal inviscid velocity ratio $= \alpha_2/\alpha_1$
Nu = Nusselt number $= q_s Pr_s x_1/(h_\infty - h_s)\mu_s$
Pr = Prandtl number, frozen
q = heat flux normal to the surface
Re = Reynolds number $= \rho_s\alpha_1 x_1^2/\mu_s$
u = component of velocity
U = airspeed
x = coordinate
α = inviscid velocity gradient, $u_i = \alpha_i x_i f_i$
λ = heat transfer parameter $= C_s g_s'/(1 - g_s)Pr_s$
μ = viscosity
η = transformed coordinate, $\eta = \left(\dfrac{\rho_\infty\alpha_1}{\mu_\infty}\right)^{1/2}\displaystyle\int_0^{x_3}(\rho/\rho_\infty)d\tilde{x}_3$
ρ = density

Subscripts

i = orthogonal coordinate system directions with 1 and 2 being along the surface and 3 normal to it
s = surface conditions
∞ = outer limit of the boundary layer

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THIS Note summarizes the results of a parametric study of heat transfer to general, regular three-dimensional stagnation points in equilibrium air flows at speeds up to 29,000 fps. The boundary-layer flows considered here are found at stagnation points of a class of three-dimensional bodies ranging from spheres, through cylinders, to saddle shapes with equal magnitudes of inviscid velocity gradients in the two principal planes. The main objective here is to present engineering data that the previous studies of this class of problems failed to produce. The most extensive published data are those of: 1) Poots[1] who considered gases with $\rho\mu$ = constant, $\rho \propto h^{-1}$, $Pr = 1.0$; and 2) Libby[2] who extended the work of Poots to $Pr = 0.7$ and surface mass transfer.

In both these studies it seems that the methods of solution influenced the decision to employ the simplifying assumption of $\rho\mu$ = const, thus, in effect, eliminating a critical heat-transfer variable and therefore producing data of limited engineering value. On the other hand Reshotko[3] obtained useful engineering estimates by employing a transformation that enabled him to use existing two-dimensional results. By developing simple, accurate engineering relations the present effort follows the pioneering work of Fay and Riddell,[4] who presented an expression for real-gas heat transfer to an axisymmetric stagnation point, and the later work of Cohen,[5] who developed correlation functions for a variety of laminar boundary-layer flows.

Governing Equations and Method of Solution

The basic differential equations describing compressible variable-properties three-dimensional boundary-layer flows have been discussed thoroughly elsewhere (e.g., Chan[6]) so that only their final form will be shown here:

$$(Cf_1'')' + (f_1 + Kf_2)f_1'' + (\rho_\infty/\rho - f_1'^2) = 0 \quad (1a)$$

$$(Cf_2'')' + (f_1 + Kf_2)f_2'' + K(\rho_\infty/\rho - f_2'^2) = 0 \quad (1b)$$

$$(Cg'/Pr)' + (f_1 + Kf_2)g' = 0 \quad (1c)$$

Table 1 Heat-transfer parameter λ

Airspeed, fps		K					
	g_s	1.0	0.6	0.2	0.0	−0.3	−0.5
5×10^3	0.30	0.996	0.892	0.784	0.730	0.666	0.663
5	0.50	0.981	0.880	0.775	0.724	0.668	0.669
5	0.70	0.965	0.865	0.763	0.715	0.668	0.670
5	0.90	0.950	0.852	0.753	0.707	0.665	0.668
10×10^3	0.10	0.986	0.884	0.776	0.723	0.658	0.655
10	0.20	0.960	0.860	0.756	0.706	0.648	0.647
10	0.30	0.939	0.842	0.741	0.693	0.639	0.640
10	0.50	0.919	0.824	0.727	0.681	0.632	0.636
15×10^3	0.05	0.990	0.887	0.779	0.725	0.661	0.657
15	0.10	0.973	0.873	0.767	0.715	0.654	0.651
15	0.20	0.956	0.857	0.754	0.704	0.647	0.647
15	0.30	0.947	0.850	0.748	0.600	0.645	0.647
20×10^3	0.05	0.995	0.892	0.783	0.730	0.666	0.661
20	0.10	0.984	0.882	0.775	0.722	0.661	0.658
20	0.20	0.974	0.874	0.768	0.717	0.659	0.658
20	0.30	0.969	0.869	0.765	0.715	0.660	0.660
25×10^3	0.01	1.020	0.914	0.802	0.747	0.679	0.673
25	0.05	1.006	0.902	0.764	0.738	0.673	0.668
25	0.10	1.000	0.896	0.788	0.734	0.671	0.668
25	0.20	0.993	0.891	0.783	0.731	0.672	0.669
25	0.30	0.989	0.887	0.781	0.729	0.672	0.672
29×10^3	0.01	1.026	0.920	0.807	0.752	0.684	0.677
29	0.05	1.015	0.910	0.799	0.744	0.674	0.672
29	0.10	1.010	0.905	0.795	0.741	0.678	0.674
29	0.20	1.004	0.900	0.791	0.739	0.678	0.676
29	0.30	0.999	0.896	0.788	0.737	0.679	0.679